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(α', β')-Derivation on the Polynomial Ring $\mathcal{K}[x]$

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Abstract Derivations are important tools in the study of algebraic structures, providing a framework for analyzing ring and module behavior through differentiation-like operations. Among their generalizations, (α, β)-derivations, defined via ring endomorphisms α and β offer increased flexibility, particularly in non-commutative settings. While these derivations have been studied extensively on rings, their behavior on polynomial extensions remains unexplored. In this paper, we investigate (α', β')-derivations on the polynomial ring $\mathcal{K}[x]$, where $\mathcal{K}$ is a ring equipped with a given (α, β)-derivation. We propose a method to construct a (α', β')-derivation on $\mathcal{K}[x]$ from the original derivation on $\mathcal{K}$, establish several of its fundamental properties, and analyze the relationship between the structures on $\mathcal{K}$ and $\mathcal{K}[x]$. Illustrative examples are provided to support the theoretical developments. This study offers a new perspective on the extension of generalized derivations to polynomial rings and contributes to the broader understanding of differential structures in algebra.

Keywords Ring, polynomial ring, derivation, (α, β)-derivation, endomorphism.

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1 Introduction

Derivations play a fundamental role in the study of algebraic structures and have been extensively explored in both commutative and non-commutative ring theory. A derivation on a ring $\mathcal{K}$ is an additive map $\Delta : \mathcal{K} \to \mathcal{K}$ that satisfies the Leibniz rule, $\Delta(\nu\mu) = \Delta(\nu)\mu + \nu\Delta(\mu)$ for every $\nu, \mu \in \mathcal{K}$, mirroring the behavior of differentiation in calculus. Among various generalizations of this concept is the (α, β)-derivation, where $\alpha, \beta : \mathcal{K} \to \mathcal{K}$ are ring endomorphisms, and the map Δ satisfies the identity $\Delta(\nu\mu) = \Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu)$, for all $\nu, \mu \in \mathcal{K}$. This generalized form provides greater flexibility for modeling differential-like behavior, particularly in algebraic contexts where ordinary derivations may fail to capture the desired structural properties.

The investigation of derivations in various algebraic settings remains a prominent focus within mathematical research. Numerous scholars have examined derivations in diverse structures, including various classes of rings and modules. In the realm of ring theory, Guven [1] explored specific types of derivation in prime rings, a study that was later expanded by Golbasi and Koc [2] through their work on Lie ideals. Ali et al. [3] analyzed how derivation maps affect commutativity in prime and semiprime rings, with further generalizations introduced by Ali and Alhazmi [4]. Based on these findings, Atteya [5] investigated the commutativity of the derivation within semi-prime rings.

Belkadi and collaborators contributed significantly by examining nilpotent homoderivations [6] and n-Jordan homoderivations [7] in prime rings. Homoderivations in semiprimary rings were studied by El-Sayiad et al. [8], while El-Deken and El-Soufi [9] explored derivation bindings, later developing them into more generalized homoderivation structures [10]. Research in this area continues to grow, with Thomas et al. [11] studying various aspects of derivations across different ring types, and Ezzat [12] introducing the concept of higher-order derivations. More recently, Gouda and Nabiel [13] extended the theory to left derivations in 2024.

In the context of modules, Bracic [14] investigated derivations and their representations, while Gurjar and Patra [15] focused on the minimal generators of derivations in modules. Considerable work has also been done on commuting derivations. Retert [16] studied them in simple rings, followed by Chen and Wang [17], who applied similar concepts to Lie algebras. Maubach [18] proposed a conjecture concerning commuting derivations in rings, Pogudin [19] explored them in fields, and Fitriani et al. [20] examined both commuting and centralizing derivations in modules. Furthermore, Fitriani et al. [21] researched f-derivations in polynomial modules, where derivations on rings formed the basis for extending derivational structures to more complex module settings.

Motivated by the work of Fitriani et al. [21], this paper investigates (α, β)-derivations on polynomial rings, a topic that has not been previously explored in the literature. We begin by examining how a (α', β')-derivation on the polynomial ring $\mathcal{K}[x]$ can be constructed from a given (α, β)-derivation on the ring $\mathcal{K}$. We then establish several properties of (α, β)-derivations on $\mathcal{K}[x]$, analyze the relationship between the derivations on $\mathcal{K}$ and those on $\mathcal{K}[x]$, and provide illustrative examples to demonstrate and support the theoretical results.

2 Methodology

This study adopts a theoretical and constructive approach to investigate (α, β)-derivations on polynomial rings. Let $\mathcal{K}$ be a ring equipped with an (α, β)-derivation $\Delta : \mathcal{K} \to \mathcal{K}$, where $\alpha, \beta : \mathcal{K} \to \mathcal{K}$ are ring endomorphisms. The core objective is to define and analyze a corresponding derivation $\Delta' : \mathcal{K}[x] \to \mathcal{K}[x]$ on the polynomial ring $\mathcal{K}[x]$, such that Δ' satisfies an (α', β')-Leibniz identity, where α', β' are suitable extensions of α, β to $\mathcal{K}[x]$.

The methodology proceeds as follows:

1. Given (α, β)-derivation on ring $\mathcal{K}$. Then, we construct α' and β' on $\mathcal{K}[x]$ by defining the extended endomorphisms $\alpha', \beta' : \mathcal{K}[x] \to \mathcal{K}[x]$ as follows:

$$\alpha'\left(\sum_{j=0}^{t} \nu_j x^j\right) = \sum_{j=0}^{t} \alpha(\nu_j) x^j, \quad \beta'\left(\sum_{j=0}^{t} \nu_j x^j\right) = \sum_{j=0}^{t} \beta(\nu_j) x^j,$$

   for every $\sum_{j=0}^{t} \nu_j x^j \in \mathcal{K}[x]$. After that, we prove that both maps are endomorphisms of $\mathcal{K}[x]$.

2. We define the extended derivation Δ' on $\mathcal{K}[x]$ by using the structure of Δ on $\mathcal{K}$ as follows:

$$\Delta'\left(\sum_{j=0}^{t} \nu_j x^j\right) = \sum_{j=0}^{t} \Delta(\nu_j) x^j,$$

   for every $\sum_{j=0}^{t} \nu_j x^j \in \mathcal{K}[x]$.
   Then, we examine whether this map satisfies the (α', β')-derivation axiom:

$$\Delta'(\lambda\zeta) = \Delta'(\lambda)\alpha'(\zeta) + \beta'(\lambda)\Delta'(\zeta), \quad \text{for all } \lambda = \sum_{j=0}^{t} \nu_j x^j, \zeta = \sum_{j=0}^{t} \mu_j x^j \in \mathcal{K}[x].$$

3. We verify the derivation properties. We rigorously prove that Δ' is an (α', β')-derivation of $R[x]$ under the defined operations, relying on the distributive and associative properties of $R[x]$ and the derivation properties of d on R.

4. We examine the structural connections between the (α, β)-derivation on $\mathcal{K}$ and its extension to the polynomial ring $\mathcal{K}[x]$. We explore how the algebraic properties of the ring $\mathcal{K}$ and the maps α, β, d influence the behavior of the corresponding structures on $\mathcal{K}[x]$.

5. We provide explicit examples of rings $\mathcal{K}$ with given (α, β)-derivations and construct the corresponding (α', β')-derivations on $\mathcal{K}[x]$ to demonstrate and validate the theoretical framework.

This approach enables a systematic extension of derivational structures from rings to their polynomial counterparts, offering new insights into generalized derivations in algebra. We begin with the definitions of derivation and (α, β)-derivation.

Definition 2.1. [22] Let $\mathcal{K}$ be a ring. An additive mapping $\Delta : \mathcal{K} \to \mathcal{K}$ is called a derivation on the ring $\mathcal{K}$ if it satisfies the following properties:

1. $\Delta(\nu + \mu) = \Delta(\nu) + \Delta(\mu)$;

2. $\Delta(\nu\mu) = \Delta(\nu)\mu + \nu\Delta(\mu)$, for all $\nu, \mu \in \mathcal{K}$.

Definition 2.2. [22] Let $\mathcal{K}$ be a ring, and let α and β be two endomorphisms on $\mathcal{K}$. An additive mapping $\Delta : \mathcal{K} \to \mathcal{K}$ is called an (α, β)-derivation if it satisfies the following condition:

$$\Delta(\nu\mu) = \Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu), \text{ for all } \nu, \mu \in \mathcal{K}.$$

3 Results and Discussion

We begin with the relationship between (α, β)-derivation and derivation on ring $\mathcal{K}$ as follows.

Proposition 3.1. *Let $\mathcal{K}$ be a ring. Every derivation is an (α, β)-derivation on $\mathcal{K}$ where α and β are the identity mappings on $\mathcal{K}$.*

Proof. Suppose that Δ is an (α, β)-derivation. Then Δ is additive and satisfies

$$\Delta(\nu\mu) = \Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu).$$

If we take $\alpha = \beta = \mathrm{id}_{\mathcal{K}}$, then

$$\Delta(\nu\mu) = \Delta(\nu)\mu + \nu\Delta(\mu),$$

which shows that Δ is a derivation on $\mathcal{K}$. □

We proceed by defining the (α, β)-derivation on the polynomial ring $\mathcal{K}[x]$ and providing an example to illustrate the construction.

Definition 3.1. Let $\mathcal{K}[x]$ be a polynomial ring, and let $\alpha', \beta' : \mathcal{K}[x] \to \mathcal{K}[x]$ be two endomorphisms on $\mathcal{K}[x]$. An additive map $\Delta' : \mathcal{K}[x] \to \mathcal{K}[x]$ is called an (α', β')-derivation if it satisfies:

$$\Delta'(\lambda\zeta) = \Delta'(\lambda)\alpha'(\zeta) + \beta'(\lambda)\Delta'(\zeta),$$

for all $\lambda, \zeta \in \mathcal{K}[x]$.

Example 3.1. Let $M_2(\mathcal{K}[x])$ be the ring of 2×2 matrices with entries in $\mathcal{K}[x]$. Define a function $\Delta' : M_2(\mathcal{K}[x]) \to M_2(\mathcal{K}[x])$ and two endomorphisms $\alpha', \beta' : M_2(\mathcal{K}[x]) \to M_2(\mathcal{K}[x])$ as follows:

$$\alpha'(\Gamma) = \begin{bmatrix} \gamma & -\chi \\ -\zeta & \lambda \end{bmatrix};$$
$$\beta'(\Gamma) = \begin{bmatrix} \gamma & \chi \\ \zeta & \lambda \end{bmatrix};$$
$$\Delta'(\Gamma) = \alpha'(\Gamma) - \beta'(\Gamma),$$

for every $\Gamma = \begin{bmatrix} \lambda & \zeta \\ \chi & \gamma \end{bmatrix} \in M_2(\mathcal{K}[x])$; where $\lambda = \sum_{j=0}^{t} \nu_j x^j, \zeta = \sum_{j=0}^{t} \mu_j x^j, \chi = \sum_{j=0}^{t} \eta_j x^j, \gamma = \sum_{j=0}^{t} \delta_j x^j \in \mathcal{K}[x]$.
We will show that Δ' is an (α', β')-derivation.

1. We will show that α' is an endomorphism on $M_2(\mathcal{K}[x])$.
   Let $\Gamma = \begin{bmatrix} \lambda & \zeta \\ \chi & \gamma \end{bmatrix}$ and $\Gamma' = \begin{bmatrix} \lambda' & \zeta' \\ \chi' & \gamma' \end{bmatrix} \in M_2(\mathcal{K}[x])$; where $\lambda = \sum_{j=0}^{t} \nu_j x^j, \zeta = \sum_{j=0}^{t} \mu_j x^j, \chi = \sum_{j=0}^{t} \eta_j x^j, \gamma = \sum_{j=0}^{t} \delta_j x^j$; and $\lambda' = \sum_{j=0}^{t} \nu'_j x^j, \zeta' = \sum_{j=0}^{t} \mu'_j x^j, \chi' = \sum_{j=0}^{t} \eta'_j x^j, \gamma' = \sum_{j=0}^{t} \delta'_j x^j \in \mathcal{K}[x]$.
   Then, the following holds:

$$\begin{aligned}
\alpha'(\Gamma + \Gamma') &= \alpha'\left(\begin{bmatrix} \lambda & \zeta \\ \chi & \gamma \end{bmatrix} + \begin{bmatrix} \lambda' & \zeta' \\ \chi' & \gamma' \end{bmatrix}\right) \\
&= \alpha'\left(\begin{bmatrix} \lambda + \lambda' & \zeta + \zeta' \\ \chi + \chi' & \gamma + \gamma' \end{bmatrix}\right) \\
&= \begin{bmatrix} \gamma + \gamma' & -(\chi + \chi') \\ -(\zeta + \zeta') & \lambda + \lambda' \end{bmatrix} \\
&= \begin{bmatrix} \gamma + \gamma' & -\chi - \chi' \\ -\zeta - \zeta' & \lambda + \lambda' \end{bmatrix} \\
&= \begin{bmatrix} \gamma & -\chi \\ -\zeta & \lambda \end{bmatrix} + \begin{bmatrix} \gamma' & -\chi' \\ -\zeta' & \lambda' \end{bmatrix} \\
&= \alpha'\left(\begin{bmatrix} \lambda & \zeta \\ \chi & \gamma \end{bmatrix}\right) + \alpha'\left(\begin{bmatrix} \lambda' & \zeta' \\ \chi' & \gamma' \end{bmatrix}\right) \\
&= \alpha'(\Gamma) + \alpha'(\Gamma'),
\end{aligned}$$

and

$$\begin{aligned}
\alpha'(\Gamma\Gamma') &= \alpha'\left(\begin{bmatrix}\lambda & \zeta\\ \chi & \gamma\end{bmatrix}\cdot\begin{bmatrix}\lambda' & \zeta'\\ \chi' & \gamma'\end{bmatrix}\right)\\
&= \alpha'\left(\begin{bmatrix}\lambda\lambda'+\zeta\chi' & \lambda\zeta'+\zeta\gamma'\\ \chi\lambda'+\gamma\chi' & \chi\zeta'+\gamma\gamma'\end{bmatrix}\right)\\
&= \begin{bmatrix}\chi\zeta'+\gamma\gamma' & -(\chi\lambda'+\gamma\chi')\\ -(\lambda\zeta'+\zeta\gamma') & \lambda\lambda'+\zeta\chi'\end{bmatrix}\\
&= \begin{bmatrix}\chi\zeta'+\gamma\gamma' & -\chi\lambda'-\gamma\chi'\\ -\lambda\zeta'-\zeta\gamma' & \lambda\lambda'+\zeta\chi'\end{bmatrix}\\
&= \begin{bmatrix}\gamma\gamma'+(-\chi)(-\zeta') & \gamma(-\chi')+(-\chi)\lambda'\\ (-\zeta)\gamma'+\lambda(-\zeta') & (-\zeta)(-\chi')+\lambda\lambda'\end{bmatrix}\\
&= \begin{bmatrix}\gamma & -\chi\\ -\zeta & \lambda\end{bmatrix}\cdot\begin{bmatrix}\gamma' & -\chi'\\ -\zeta' & \lambda'\end{bmatrix}\\
&= \alpha'\left(\begin{bmatrix}\lambda & \zeta\\ \chi & \gamma\end{bmatrix}\right)\cdot\alpha'\left(\begin{bmatrix}\lambda' & \zeta'\\ \chi' & \gamma'\end{bmatrix}\right)\\
&= \alpha(\Gamma)\alpha(\Gamma').
\end{aligned}$$

It has been proven that α' is an endomorphism on $M_2(\mathcal{K}[x])$.

2. We will show that β' is an endomorphism on $M_2(\mathcal{K}[x])$.

   Let $\Gamma = \begin{bmatrix}\lambda & \zeta\\ \chi & \gamma\end{bmatrix}$ and $\Gamma' = \begin{bmatrix}\lambda' & \zeta'\\ \chi' & \gamma'\end{bmatrix} \in M_2(\mathcal{K}[x])$. Then, the following holds:

$$\begin{aligned}
\beta'(\Gamma+\Gamma') &= \beta'\left(\begin{bmatrix}\lambda & \zeta\\ \chi & \gamma\end{bmatrix}+\begin{bmatrix}\lambda' & \zeta'\\ \chi' & \gamma'\end{bmatrix}\right)\\
&= \beta'\left(\begin{bmatrix}\lambda+\lambda' & \zeta+\zeta'\\ \chi+\chi' & \gamma+\gamma'\end{bmatrix}\right)\\
&= \begin{bmatrix}\gamma+\gamma' & \chi+\chi'\\ \zeta+\zeta' & \lambda+\lambda'\end{bmatrix}\\
&= \begin{bmatrix}\gamma & \chi\\ \zeta & \lambda\end{bmatrix}+\begin{bmatrix}\gamma' & \chi'\\ \zeta' & \lambda'\end{bmatrix}\\
&= \beta'\left(\begin{bmatrix}\lambda & \zeta\\ \chi & \gamma\end{bmatrix}\right)+\beta'\left(\begin{bmatrix}\lambda' & \zeta'\\ \chi' & \gamma'\end{bmatrix}\right)\\
&= \beta'(\Gamma)+\beta'(\Gamma'),
\end{aligned}$$

and

$$\begin{aligned}
\beta'(\Gamma\Gamma') &= \beta'\left(\begin{bmatrix}\lambda & \zeta\\ \chi & \gamma\end{bmatrix}\cdot\begin{bmatrix}\lambda' & \zeta'\\ \chi' & \gamma'\end{bmatrix}\right)\\
&= \beta'\left(\begin{bmatrix}\lambda\lambda'+\zeta\chi' & \lambda\zeta'+\zeta\gamma'\\ \chi\lambda'+\gamma\chi' & \chi\zeta'+\gamma\gamma'\end{bmatrix}\right)\\
&= \begin{bmatrix}\chi\zeta'+\gamma\gamma' & \chi\lambda'+\gamma\chi'\\ \lambda\zeta'+\zeta\gamma' & \lambda\lambda'+\zeta\chi'\end{bmatrix}\\
&= \begin{bmatrix}\gamma\gamma'+\chi\zeta' & \gamma\chi'+\chi\lambda'\\ \zeta\gamma'+\lambda\zeta' & \zeta\chi'+\lambda\lambda'\end{bmatrix}\\
&= \begin{bmatrix}\gamma & \chi\\ \zeta & \lambda\end{bmatrix}\cdot\begin{bmatrix}\gamma' & \chi'\\ \zeta' & \lambda'\end{bmatrix}\\
&= \beta'\left(\begin{bmatrix}\lambda & \zeta\\ \chi & \gamma\end{bmatrix}\right)\cdot\beta'\left(\begin{bmatrix}\lambda' & \zeta'\\ \chi' & \gamma'\end{bmatrix}\right)\\
&= \beta(\Gamma)\beta(\Gamma').
\end{aligned}$$

It has been proven that β' is an endomorphism on $M_2(\mathcal{K}[x])$.

3. We will show that d is an (α', β')-derivation on $M_2(\mathcal{K}[x])$.
Let $\Gamma = \begin{bmatrix} \lambda & \zeta \\ \chi & \gamma \end{bmatrix}$ and $\Gamma' = \begin{bmatrix} \lambda' & \zeta' \\ \chi' & \gamma' \end{bmatrix} \in M_2(\mathcal{K}[x])$. Then, the following holds:

$$\begin{aligned} \Delta'(\Gamma + \Gamma') &= \alpha'(\Gamma + \Gamma') - \beta'(\Gamma + \Gamma') \\ &= \alpha'(\Gamma) + \alpha'(\Gamma') - \beta'(\Gamma) - \beta'(\Gamma') \\ &= (\alpha'(\Gamma) - \beta'(\Gamma)) + (\alpha'(\Gamma') - \beta'(\Gamma')) \\ &= \Delta'(\Gamma) + \Delta'(\Gamma'), \end{aligned}$$

and

$$\begin{aligned} \Delta'(\Gamma\Gamma') &= \alpha'(\Gamma\Gamma') - \beta'(\Gamma\Gamma') \\ &= \alpha'(\Gamma)\alpha'(\Gamma') - \beta'(\Gamma)\beta'(\Gamma') \\ &= \alpha'(\Gamma)\alpha'(\Gamma') - \beta'(\Gamma)\alpha'(\Gamma') + \beta'(\Gamma)\alpha'(\Gamma') - \beta'(\Gamma)\beta'(\Gamma') \\ &= (\alpha'(\Gamma) - \beta'(\Gamma))\,\alpha'(\Gamma') + \beta'(\Gamma)\,(\alpha'(\Gamma') - \beta'(\Gamma')) \\ &= \Delta'(\Gamma)\alpha'(\Gamma') + \beta'(\Gamma)\Delta'(\Gamma') \end{aligned}$$

Since all the axioms are satisfied, it is proven that d is an (α', β')-derivation on $M_2(\mathcal{K}[x])$.

In the following lemma, we establish that an additive mapping on the polynomial ring $\mathcal{K}[x]$ can be naturally induced by an additive mapping on the base ring $\mathcal{K}$.

Lemma 3.2. *Let $\mathcal{K}$ be a ring. If $\Delta : \mathcal{K} \to \mathcal{K}$ is an additive map on $\mathcal{K}$, then there exists a map $\Delta' : \mathcal{K}[x] \to \mathcal{K}[x]$ which is an additive map on $\mathcal{K}[x]$.*

Proof. Define $\Delta' : \mathcal{K}[x] \to \mathcal{K}[x]$ as:

$$\Delta' \left(\sum_{j=0}^{t} \nu_j x^j \right) = \sum_{j=0}^{t} \Delta(\nu_j) x^j$$

for each $\nu_j \in \mathcal{K}$. We will show that Δ' is an additive map on $\mathcal{K}[x]$.
Let $\lambda = \sum_{i=0}^{t} \nu_i x^i, \zeta = \sum_{j=0}^{s} \mu_j x^j \in \mathcal{K}[x]$ be arbitrary. The following holds:

$$\begin{aligned} \Delta'(\lambda + \zeta) &= \Delta' \left(\sum_{i=0}^{t} \nu_i x^i + \sum_{j=0}^{s} \mu_j x^j \right) \\ &= \Delta' \left(\sum_{k=0}^{\max(t,s)} (\nu_k + \mu_k) x^k \right) \\ &= \sum_{k=0}^{\max(t,s)} \Delta(\nu_k + \mu_k) x^k. \end{aligned}$$

Since Δ is additive on $\mathcal{K}$, i.e., $\Delta(\nu_k + \mu_k) = \Delta(\nu_k) + \Delta(\mu_k)$, we get:

$$\begin{aligned} \Delta'(\lambda + \zeta) &= \sum_{k=0}^{\max(t,s)} (\Delta(\nu_k) + \Delta(\mu_k))\, x^k \\ &= \sum_{k=0}^{\max(t,s)} \left(\Delta(\nu_k) x^k + \Delta(\mu_k) x^k\right) \\ &= \sum_{k=0}^{\max(t,s)} \Delta(\nu_k) x^k + \sum_{k=0}^{\max(t,s)} \Delta(\mu_k) x^k \\ &= \Delta' \left(\sum_{k=0}^{\max(t,s)} \nu_k x^k \right) + \Delta' \left(\sum_{k=0}^{\max(t,s)} \mu_k x^k \right) \\ &= \Delta' \left(\sum_{i=0}^{t} \nu_i x^i \right) + \Delta' \left(\sum_{j=0}^{s} \mu_j x^j \right) \\ &= \Delta'(\lambda) + \Delta'(\zeta). \end{aligned}$$

Since $\Delta'(\lambda+\zeta) = \Delta'(\lambda) + \Delta'(\zeta)$, it is proven that Δ' is an additive map on $\mathcal{K}[x]$. □

In the following lemma, we demonstrate that any endomorphism on the base ring $\mathcal{K}$ induces a corresponding endomorphism on the polynomial ring $\mathcal{K}[x]$.

Lemma 3.3. *Let $\mathcal{K}$ be a ring. If $\alpha, \beta : \mathcal{K} \to \mathcal{K}$ are two endomorphisms on $\mathcal{K}$, then there exist $\alpha', \beta' : \mathcal{K}[x] \to \mathcal{K}[x]$ that are endomorphisms on $\mathcal{K}[x]$.*

Proof. Define $\alpha', \beta' : \mathcal{K}[x] \to \mathcal{K}[x]$ as:

$$\alpha'\left(\sum_{i=0}^{t} \nu_i x^i\right) = \sum_{i=0}^{t} \alpha(\nu_i) x^i;$$

$$\beta'\left(\sum_{j=0}^{s} \mu_j x^j\right) = \sum_{j=0}^{s} \beta(\mu_j) x^j,$$

for each $\nu_i, \mu_j \in \mathcal{K}$.

We will show that α' and β' are endomorphisms on $\mathcal{K}[x]$.

1. Given $\lambda = \sum_{i=0}^{t} \nu_i x^i, \zeta = \sum_{j=0}^{s} \mu_j x^j \in \mathcal{K}[x]$, we have:

$$\begin{aligned}\alpha'(\lambda+\zeta) &= \alpha'\left(\sum_{i=0}^{t} \nu_i x^i + \sum_{j=0}^{s} \mu_j x^j\right) \\ &= \alpha'\left(\sum_{k=0}^{\max(t,s)} (\nu_k + \mu_k) x^k\right) \\ &= \sum_{k=0}^{\max(t,s)} \alpha(\nu_k + \mu_k) x^k.\end{aligned}$$

Since α is an endomorphism on $\mathcal{K}$, i.e., $\alpha(\nu_k + \mu_k) = \alpha(\nu_k) + \alpha(\mu_k)$, we get:

$$\begin{aligned}\alpha'(\lambda+\zeta) &= \sum_{k=0}^{\max(t,s)} \left(\alpha(\nu_k) + \alpha(\mu_k)\right) x^k \\ &= \sum_{k=0}^{\max(t,s)} \left(\alpha(\nu_k) x^k + \alpha(\mu_k) x^k\right) \\ &= \sum_{k=0}^{\max(t,s)} \alpha(\nu_k) x^k + \sum_{k=0}^{\max(t,s)} \alpha(\mu_k) x^k \\ &= \alpha'\left(\sum_{k=0}^{\max(t,s)} \nu_k x^k\right) + \alpha'\left(\sum_{k=0}^{\max(t,s)} \mu_k x^k\right) \\ &= \alpha'\left(\sum_{i=0}^{t} \nu_i x^i\right) + \alpha'\left(\sum_{j=0}^{s} \mu_j x^j\right) \\ &= \alpha'(\lambda) + \alpha'(\zeta).\end{aligned}$$

Similarly:

$$\begin{aligned}\beta'(\lambda+\zeta) &= \beta'\left(\sum_{i=0}^{t} \nu_i x^i + \sum_{j=0}^{s} \mu_j x^j\right) \\ &= \beta'\left(\sum_{k=0}^{\max(t,s)} (\nu_k + \mu_k) x^k\right) \\ &= \sum_{k=0}^{\max(t,s)} \beta(\nu_k + \mu_k) x^k.\end{aligned}$$

Since β is an endomorphism on $\mathcal{K}$, i.e., $\beta(\nu_k + \mu_k) = \beta(\nu_k) + \beta(\mu_k)$, we get:

$$\begin{aligned}
\beta'(\lambda+\zeta) &= \sum_{k=0}^{\max(t,s)} \left(\beta(\nu_k) + \beta(\mu_k)\right) x^k \\
&= \sum_{k=0}^{\max(t,s)} \left(\beta(\nu_k)x^k + \beta(\mu_k)x^k\right) \\
&= \sum_{k=0}^{\max(t,s)} \beta(\nu_k)x^k + \sum_{k=0}^{\max(t,s)} \beta(\mu_k)x^k \\
&= \beta'\left(\sum_{k=0}^{\max(t,s)} \nu_k x^k\right) + \beta'\left(\sum_{k=0}^{\max(t,s)} \mu_k x^k\right) \\
&= \beta'\left(\sum_{i=0}^{t} \nu_i x^i\right) + \beta'\left(\sum_{j=0}^{s} \mu_j x^j\right) \\
&= \beta'(\lambda) + \beta'(\zeta).
\end{aligned}$$

Therefore, $\alpha'(\lambda+\zeta) = \alpha'(\lambda) + \alpha'(\zeta)$ and $\beta'(\lambda+\zeta) = \beta'(\lambda) + \beta'(\zeta)$.

2. Given $\lambda = \sum_{i=0}^{t} \nu_i x^i, \zeta = \sum_{j=0}^{s} \mu_j x^j \in \mathcal{K}[x]$, we have:

$$\begin{aligned}
\alpha'(\lambda\zeta) &= \alpha'\left(\left(\sum_{i=0}^{t} \nu_i x^i\right)\left(\sum_{j=0}^{s} \mu_j x^j\right)\right) \\
&= \alpha'\left(\sum_{k=0}^{t+s}\left(\sum_{i+j=k} \nu_i\mu_j\right) x^k\right) \\
&= \sum_{k=0}^{t+s} \alpha\left(\sum_{i+j=k} \nu_i\mu_j\right) x^k.
\end{aligned}$$

Since α is an endomorphism on $\mathcal{K}$, i.e., $\alpha\left(\sum_{i+j=k} \nu_i\mu_j\right) = \sum_{i+j=k} \alpha(\nu_i)\alpha(\mu_j)$, we get:

$$\begin{aligned}
\alpha'(\lambda\zeta) &= \sum_{k=0}^{t+s}\left(\sum_{i+j=k} \alpha(\nu_i)\alpha(\mu_j)\right) x^k \\
&= \left(\sum_{i=0}^{t} \alpha(\nu_i)x^i\right)\left(\sum_{j=0}^{s} \alpha(\mu_j)x^j\right) \\
&= \alpha'\left(\sum_{i=0}^{t} \nu_i x^i\right) \alpha'\left(\sum_{j=0}^{s} \mu_j x^j\right) \\
&= \alpha'(\lambda)\alpha'(\zeta).
\end{aligned}$$

Similarly:

$$\begin{aligned}
\beta'(\lambda\zeta) &= \beta'\left(\left(\sum_{i=0}^{t} \nu_i x^i\right)\left(\sum_{j=0}^{s} \mu_j x^j\right)\right) \\
&= \beta'\left(\sum_{k=0}^{t+s}\left(\sum_{i+j=k} \nu_i\mu_j\right) x^k\right) \\
&= \sum_{k=0}^{t+s} \beta\left(\sum_{i+j=k} \nu_i\mu_j\right) x^k.
\end{aligned}$$

Since β is an endomorphism on $\mathcal{K}$, i.e., $\beta\left(\sum_{i+j=k} \nu_i \mu_j\right) = \sum_{i+j=k} \beta(\nu_i)\beta(\mu_j)$, we get:

$$\begin{aligned}\beta'(\lambda\zeta) &= \sum_{k=0}^{t+s}\left(\sum_{i+j=k} \alpha(\nu_i)\beta(\mu_j)\right)x^k \\ &= \left(\sum_{i=0}^{t} \beta(\nu_i)x^i\right)\left(\sum_{j=0}^{s} \beta(\mu_j)x^j\right) \\ &= \beta'\left(\sum_{i=0}^{t} \nu_i x^i\right)\beta'\left(\sum_{j=0}^{s} \mu_j x^j\right) \\ &= \beta'(\lambda)\beta'(\zeta).\end{aligned}$$

Therefore, $\alpha'(\lambda\zeta) = \alpha'(\lambda)\alpha'(\zeta)$ and $\beta'(\lambda\zeta) = \beta'(\lambda)\beta'(\zeta)$.

Since all the axioms of an endomorphism are satisfied, it is proven that α' and β' are endomorphisms on $\mathcal{K}[x]$. □

The following theorem establishes that an (α, β)-derivation on $\mathcal{K}$ can be naturally extended to a (α', β')-derivation on $\mathcal{K}[x]$.

Theorem 3.4. *Let $\mathcal{K}$ be a ring. If $\Delta : \mathcal{K} \to \mathcal{K}$ is an (α, β)-derivation on $\mathcal{K}$, where $\alpha, \beta : \mathcal{K} \to \mathcal{K}$ are endomorphisms on $\mathcal{K}$, then there exists a map $\Delta' : \mathcal{K}[x] \to \mathcal{K}[x]$ which is an (α', β')-derivation on $\mathcal{K}[x]$, where $\alpha', \beta' : \mathcal{K}[x] \to \mathcal{K}[x]$ are endomorphisms on $\mathcal{K}[x]$.*

Proof. We define:

$$\Delta'\left(\sum_{i=0}^{t} \nu_i x^i\right) = \sum_{i=0}^{t} \Delta(\nu_i)x^i,$$

$$\alpha'\left(\sum_{i=0}^{t} \nu_i x^i\right) = \sum_{i=0}^{t} \alpha(\nu_i)x^i,$$

and

$$\beta'\left(\sum_{i=0}^{t} \nu_i x^i\right) = \sum_{i=0}^{t} \beta(\nu_i)x^i,$$

for every $\sum_{j=0}^{t} \nu_j x^j \in \mathcal{K}[x]$.

Based on Lemma 3.2 and Lemma 3.3, we have Δ' is an additive map, and α', β' are endomorphisms on $\mathcal{K}[x]$.

We will show that Δ' is an (α', β')-derivation.

Given arbitrary $\lambda = \sum_{i=0}^{t} \nu_i x^i, \zeta = \sum_{j=0}^{s} \mu_j x^j \in \mathcal{K}[x]$, we compute:

$$\Delta'(\lambda\zeta) = \Delta'\left(\left(\sum_{i=0}^{t} \nu_i x^i\right)\left(\sum_{j=0}^{s} \mu_j x^j\right)\right) \tag{1}$$

$$= \Delta'\left(\sum_{k=0}^{t+s}\left(\sum_{i+j=k} \nu_i \mu_j\right)x^k\right) \tag{2}$$

$$= \sum_{k=0}^{t+s} \Delta\left(\sum_{i+j=k} \nu_i \mu_j\right)x^k. \tag{3}$$

$$= \sum_{k=0}^{t+s}\left(\sum_{i+j=k} \Delta(\nu_i \mu_j)\right)x^k. \tag{4}$$

Since Δ is an (α, β)-derivation on $\mathcal{K}$, it satisfies:

$$\Delta(\nu_i \mu_j) = \Delta(\nu_i)\alpha(\mu_j) + \beta(\nu_i)\Delta(\mu_j). \tag{5}$$

Substituting Equation (5) into Equation (4), we obtain:

$$\begin{aligned}
\Delta'(\lambda\zeta) &= \sum_{k=0}^{t+s}\left(\sum_{i+j=k}(\Delta(\nu_i)\alpha(\mu_j)+\beta(\nu_i)\Delta(\mu_j))\right)x^k \\
&= \sum_{k=0}^{t+s}\left(\sum_{i+j=k}\Delta(\nu_i)\alpha(\mu_j)+\sum_{i+j=k}\beta(\nu_i)\Delta(\mu_j)\right)x^k \\
&= \sum_{k=0}^{t+s}\left(\sum_{i+j=k}\Delta(\nu_i)\alpha(\mu_j)\right)x^k+\sum_{k=0}^{t+s}\left(\sum_{i+j=k}\beta(\nu_i)\Delta(\mu_j)\right)x^k. \\
&= \left(\sum_{i=0}^{t}\Delta(\nu_i)x^i\right)\left(\sum_{j=0}^{s}\alpha(\mu_j)x^j\right)+\left(\sum_{i=0}^{t}\beta(\nu_i)x^i\right)\left(\sum_{j=0}^{s}\Delta(\mu_j)x^j\right) \\
&= \Delta'\left(\sum_{i=0}^{t}\nu_i x^i\right)\alpha'\left(\sum_{j=0}^{s}\mu_j x^j\right)+\beta'\left(\sum_{i=0}^{t}\nu_i x^i\right)\Delta'\left(\sum_{j=0}^{s}\mu_j x^j\right) \\
&= \Delta'(\lambda)\alpha'(\zeta)+\beta'(\lambda)\Delta'(\zeta)
\end{aligned}$$

Since $\Delta'(\lambda\zeta) = \Delta'(\lambda)\alpha'(\zeta) + \beta'(\lambda)\Delta'(\zeta)$, it follows that $\Delta' : \mathcal{K}[x] \to \mathcal{K}[x]$ is an (α', β')-derivation. □

If Δ_1, Δ_2 are (α, β)-derivations, then the composition $\Delta_1 \circ \Delta_2$ is not necessarily an (α, β)-derivation. This indicates that the structure of an (α, β)-derivation possesses unique characteristics that distinguish it from an ordinary derivation. The following is an example of the composition of two (α, β)-derivations which does not satisfy the properties of an (α, β)-derivation.

Example 3.2. Given the ring $\mathcal{K} = \mathbb{R}[x]$, the polynomial ring with real coefficients, and let $\Delta_1, \Delta_2 : \mathcal{K} \to \mathcal{K}$ be (α, β)-derivations with $\alpha, \beta : \mathcal{K} \to \mathcal{K}$ endomorphisms on $\mathcal{K}$ defined as the identity mappings. We will show that $\Delta_1 \circ \Delta_2$ is not an (α, β)-derivation. Define:

$$\begin{aligned}
\Delta_1(\rho(x)) &= \rho'(x); \\
\Delta_2(\rho(x)) &= \rho(x); \\
\alpha(\rho(x)) &= \rho(x); \\
\beta(\rho(x)) &= \rho(x), \quad \text{for every } \rho(x) \in \mathcal{K}.
\end{aligned}$$

Choose $\rho(x) = 2x^2 + 1$ and $\xi(x) = 3x - 1 \in \mathcal{K}$, then:

$$\begin{aligned}
(\Delta_1 \circ \Delta_2)(\rho(x)\xi(x)) &= \Delta_1(\Delta_2(\rho(x)\xi(x))) \\
&= \Delta_1(\Delta_2(\rho(x))\alpha(\xi(x)) + \beta(\rho(x))\Delta_2(\xi(x))) \\
&= \Delta_1(\rho(x)\xi(x) + \rho(x)\xi(x)) \\
&= \Delta_1(2\rho(x)\xi(x)) \\
&= 2(\Delta_1(\rho(x))\alpha(\xi(x)) + \beta(\rho(x))\Delta_1(\xi(x))) \\
&= 2(\Delta_1(2x^2+1)\alpha(3x-1) + \beta(2x^2+1)\Delta_1(3x-1)) \\
&= 2((4x)(3x-1) + (2x^2+1)(3)) \\
&= 2(12x^2 - 4x + 6x^2 + 3) \\
&= 2(18x^2 - 4x + 3) \\
&= 36x^2 - 8x + 6.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&(\Delta_1 \circ \Delta_2)(\rho(x))\alpha(\xi(x)) + \beta(\rho(x))(\Delta_1 \circ \Delta_2)\xi(x)) \\
&= \Delta_1(\Delta_2(\rho(x)))\xi(x) + \rho(x)\Delta_1(\Delta_2(\xi(x))) \\
&= \Delta_1(2x^2+1)(3x-1) + (2x^2+1)\Delta_1(3x-1) \\
&= (4x)(3x-1) + (2x^2+1)(3) \\
&= 12x^2 - 4x + 6x^2 + 3 \\
&= 18x^2 - 4x + 3.
\end{aligned}$$

Since $(\Delta_1 \circ \Delta_2)(\rho(x)\xi(x)) \neq (\Delta_1 \circ \Delta_2)(\rho(x))\alpha(\xi(x)) + \beta(\rho(x))(\Delta_1 \circ \Delta_2)(\xi(x))$, it follows that $\Delta_1 \circ \Delta_2$ is not an (α, β)-derivation.

The following theorem demonstrates that the set of (α, β)-derivations on $\mathcal{K}$ is closed under addition.

Theorem 3.5. *Let R be a ring. If $\Delta_1, \Delta_2, \ldots, \Delta_n : \mathcal{K} \to \mathcal{K}$ are (α, β)-derivations on $\mathcal{K}$, where $(\alpha, \beta) : \mathcal{K} \to \mathcal{K}$ are endomorphisms on $\mathcal{K}$, then the sum $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_n$ is also an (α, β)-derivation.*

Proof. We will show that $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_n = \sum_{i=1}^{n} \Delta_i$ is an (α, β)-derivation.

1. Given $\nu, \mu \in \mathcal{K}$, we have:

$$\begin{aligned}
\Delta(\nu + \mu) &= \left(\sum_{i=1}^{n} \Delta_i\right)(\nu + \mu) \\
&= \sum_{i=1}^{n} (\Delta_i(\nu + \mu)) \\
&= \sum_{i=1}^{n} (\Delta_i(\nu) + \Delta_i(\mu)) \\
&= \sum_{i=1}^{n} (\Delta_i(\nu)) + \sum_{i=1}^{n} (\Delta_i(\mu)) \\
&= \left(\sum_{i=1}^{n} \Delta_i\right)(\nu) + \left(\sum_{i=1}^{n} \Delta_i\right)(\mu) \\
&= \Delta(\nu) + \Delta(\mu).
\end{aligned}$$

   Since $\Delta(\nu + \mu) = \Delta(\nu) + \Delta(\mu)$, it follows that d is additive.

2. Given $\nu, \mu \in \mathcal{K}$, we have:

$$\begin{aligned}
\Delta(\nu\mu) &= \left(\sum_{i=1}^{n} \Delta_i\right)(\nu\mu) \\
&= \sum_{i=1}^{n} (\Delta_i(\nu\mu))
\end{aligned}$$

   Since each Δ_i is an (α, β)-derivation, i.e.,

$$\Delta_i(\nu\mu) = \Delta_i(\nu)\alpha(\mu) + \beta(\nu)\Delta_i(\mu) \quad \text{for all } i = 1, 2, \ldots, n,$$

it follows that:

$$\begin{aligned}
\Delta(\nu\mu) &= \sum_{i=1}^{n}(\Delta_i(\nu)\alpha(\mu) + \beta(\nu)\Delta_i(\mu)) \\
&= \sum_{i=1}^{n}(\Delta_i(\nu)\alpha(\mu)) + \sum_{i=1}^{n}(\beta(\nu)\Delta_i(\mu)) \\
&= \sum_{i=1}^{n}(\Delta_i(\nu))\alpha(\mu) + \beta(\nu)\sum_{i=1}^{n}(\Delta_i(\mu)) \\
&= \left(\sum_{i=1}^{n}\Delta_i\right)(\nu)\alpha(\mu) + \beta(\nu)\left(\sum_{i=1}^{n}\Delta_i\right)(\mu) \\
&= \Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu).
\end{aligned}$$

Therefore, since $\Delta(\nu\mu) = \Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu)$, it is proven that $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_n$ is an (α, β)-derivation.

$\square$

The following theorem demonstrates that the set of (α, β)-derivations on $\mathcal{K}$ is closed under linear combination.

Theorem 3.6. *Let $\mathcal{K}$ be a ring. If $\Delta_1, \Delta_2 : \mathcal{K} \to \mathcal{K}$ are (α, β)-derivations on $\mathcal{K}$, where $(\alpha, \beta) : \mathcal{K} \to \mathcal{K}$ are endomorphisms on $\mathcal{K}$, then for any $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{K}$, the map*

$$\Delta = \gamma_1\Delta_1 + \gamma_2\Delta_2 + \cdots + \gamma_n\Delta_n = \sum_{i=1}^{n}\gamma_i\Delta_i$$

is also an (α, β)-derivation.

Proof. We will show that $\Delta = \sum_{i=1}^{n}\gamma_i\Delta_i$ is an (α, β)-derivation.

1. Let $\nu, \mu \in \mathcal{K}$. Then we have:

$$\begin{aligned}
\Delta(\nu + \mu) &= \left(\sum_{i=1}^{n}\gamma_i\Delta_i\right)(\nu + \mu) \\
&= \sum_{i=1}^{n}(\gamma_i\Delta_i)(\nu + \mu) \\
&= \sum_{i=1}^{n}\gamma_i(\Delta_i(\nu + \mu)) \\
&= \sum_{i=1}^{n}\gamma_i(\Delta_i(\nu) + \Delta_i(\mu)) \\
&= \sum_{i=1}^{n}(\gamma_i\Delta_i)(\nu) + \sum_{i=1}^{n}(\gamma_i\Delta_i)(\mu) \\
&= \left(\sum_{i=1}^{n}\gamma_i\Delta_i\right)(\nu) + \left(\sum_{i=1}^{n}\gamma_i\Delta_i\right)(\mu) \\
&= \Delta(\nu) + \Delta(\mu).
\end{aligned}$$

Therefore, Δ is additive.

2. Let $\nu, \mu \in \mathcal{K}$. Then:

$$\begin{aligned}
\Delta(\nu\mu) &= \left(\sum_{i=1}^{n}\gamma_i\Delta_i\right)(\nu\mu) \\
&= \sum_{i=1}^{n}(\gamma_i\Delta_i)(\nu\mu) \\
&= \sum_{i=1}^{n}\gamma_i(\Delta_i(\nu\mu)).
\end{aligned}$$

Since each Δ_i is an (α, β)-derivation, we have:

$$\Delta_i(\nu\mu) = \Delta_i(\nu)\alpha(\mu) + \beta(\nu)\Delta_i(\mu)$$

for all $i = 1, 2, \ldots, n$. Thus:

$$\begin{aligned}
\Delta(\nu\mu) &= \sum_{i=1}^{n} \gamma_i \left(\Delta_i(\nu)\alpha(\mu) + \beta(\nu)\Delta_i(\mu)\right) \\
&= \sum_{i=1}^{n} \gamma_i \left(\Delta_i(\nu)\alpha(\mu)\right) + \sum_{i=1}^{n} \gamma_i \left(\beta(\nu)\Delta_i(\mu)\right) \\
&= \sum_{i=1}^{n} \left(\gamma_i\Delta_i\right)(\nu)\alpha(\mu) + \sum_{i=1}^{n} \beta(\nu)\left(\gamma_i\Delta_i\right)(\mu) \\
&= \left(\sum_{i=1}^{n} \gamma_i\Delta_i\right)(\nu)\alpha(\mu) + \beta(\nu)\left(\sum_{i=1}^{n} \gamma_i\Delta_i\right)(\mu) \\
&= \Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu).
\end{aligned}$$

Hence, $\Delta = \gamma_1\Delta_1 + \gamma_2\Delta_2 + \cdots + \gamma_n\Delta_n = \sum_{i=1}^{n} \gamma_i\Delta_i$ is an (α, β)-derivation. □

Let $\mathcal{K}$ be a ring and let $\Delta_1, \Delta_2 : \mathcal{K} \to \mathcal{K}$. Suppose Δ_1 is an (α_1, β_1)-derivation and Δ_2 is an (α_2, β_2)-derivation, where $\alpha_1, \alpha_2, \beta_1, \beta_2 : \mathcal{K} \to \mathcal{K}$ are endomorphisms on $\mathcal{K}$. Then the map $\Delta = \Delta_1 + \Delta_2$ is not necessarily an $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$-derivation. The following is an example showing that Δ is not an $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$-derivation.

Example 3.3. Let $\mathcal{K}$ be a ring. Define $\Delta_1, \Delta_2 : \mathcal{K} \to \mathcal{K}$ such that Δ_1 is an (α_1, β_1)-derivation and Δ_2 is an (α_2, β_2)-derivation, where $\alpha_1, \alpha_2, \beta_1, \beta_2 : \mathcal{K} \to \mathcal{K}$ are endomorphisms on $\mathcal{K}$. We will investigate whether $\Delta = \Delta_1 + \Delta_2$ is not an $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$-derivation.

1. Let $\nu, \mu \in R$. We have:

$$\begin{aligned}
\Delta(\nu + \mu) &= (\Delta_1 + \Delta_2)(\nu + \mu) \\
&= (\Delta_1 + \Delta_2)(\nu) + (\Delta_1 + \Delta_2)(\mu) \\
&= \Delta(\nu) + \Delta(\mu).
\end{aligned}$$

Since $\Delta(\nu + \mu) = \Delta(\nu) + \Delta(\mu)$, Δ is additive.

2. Let $\nu, \mu \in R$. We have:

$$\begin{aligned}
\Delta(\nu\mu) &= (\Delta_1 + \Delta_2)(\nu\mu) \\
&= \Delta_1(\nu\mu) + \Delta_2(\nu\mu).
\end{aligned}$$

Since Δ_1 is an (α_1, β_1)-derivation, it follows that $\Delta_1(\nu\mu) = \Delta_1(\nu)\alpha_1(\mu) + \beta_1(\nu)\Delta_1(\mu)$, and since Δ_2 is an (α_2, β_2)-derivation, it follows that $\Delta_2(\nu\mu) = \Delta_2(\nu)\alpha_2(\mu) + \beta_2(\nu)\Delta_2(\mu)$.

We obtain:

$$\Delta(\nu\mu) = (\Delta_1(\nu)\alpha_1(\mu) + \beta_1(\nu)\Delta_1(\mu)) + (\Delta_2(\nu)\alpha_2(\mu) + \beta_2(\nu)\Delta_2(\mu)).$$

On the other hand, we have:

$$\begin{aligned}
&\Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu) \\
&= (\Delta_1 + \Delta_2)(\nu)(\alpha_1 + \alpha_2)(\mu) + (\beta_1 + \beta_2)(\nu)(\Delta_1 + \Delta_2)(\mu) \\
&= (\Delta_1(\nu) + \Delta_2(\nu))(\alpha_1(\mu) + \alpha_2(\mu)) + (\beta_1(\nu) + \beta_2(\nu))(\Delta_1(\mu) + \Delta_2(\mu)) \\
&= \Delta_1(\nu)\alpha_1(\mu) + \Delta_1(\nu)\alpha_2(\mu) + \Delta_2(\nu)\alpha_1(\mu) + \Delta_2(\nu)\alpha_2(\mu) \\
&\quad + \beta_1(\nu)\Delta_1(\mu) + \beta_1(\nu)\Delta_2(\mu) + \beta_2(\nu)\Delta_1(\mu) + \beta_2(\nu)\Delta_2(\mu) \\
&\neq \Delta(\nu\mu).
\end{aligned}$$

Since $\Delta(\nu\mu) \neq \Delta(\nu)\alpha(\mu) + \beta(\nu)\Delta(\mu)$ or $(\Delta_1 + \Delta_2)(\nu\mu) \neq (\Delta_1 + \Delta_2)(\nu)(\alpha_1 + \alpha_2)(\mu) + (\beta_1 + \beta_2)(\nu)(\Delta_1 + \Delta_2)(\mu)$, we conclude that $\Delta = \Delta_1 + \Delta_2$ is not an $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$-derivation.

4 Conclusions

From the discussion, it can be concluded that for every (α, β)-derivation on a ring $\mathcal{K}$ with α and β being endomorphisms in $\mathcal{K}$, there exists an (α', β')-derivation on the polynomial ring $\mathcal{K}[x]$ with α' and β' being endomorphisms in $\mathcal{K}[x]$. Additionally, this research shows that derivations are a special case of (α, β)-derivations where α and β are identity maps.

This research also demonstrates that the properties of (α, β)-derivations have unique characteristics, particularly in composition and combination. The composition of two (α, β)-derivations does not always result in a (α, β)-derivation, which distinguishes (α, β)-derivations from regular derivations. On the other hand, a linear combination of (α, β)-derivations still preserves the structure of (α, β)-derivations. Furthermore, this study also shows that if there are two (α_1, β_1)-derivations and (α_2, β_2)-derivations, their combination is not necessarily a $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$-derivation. This highlights certain limitations of the (α, β)-derivation operation in terms of combination and the propagation of properties within the polynomial ring $\mathcal{K}[x]$.

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