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On Sombor Index and Sombor Polynomial of Co-Prime Graphs for Finite Groups

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Abstract The Sombor polynomial is a polynomial whose power of the indeterminate is the sum of the squares of the degrees of adjacent vertices in a graph. In this paper, the graph considered is the co-prime graph of a finite group G; the graph has all elements of G as vertices with two distinct elements adjacent in the graph if and only if their orders are relatively prime. Formulae for obtaining the Sombor index and Sombor polynomial of the graphs for $p-$groups and groups of order the product of two distinct primes are derived. Two variations of the graph are also considered; the co-prime order graph and the co-prime power order graph, for groups of order the product of two distinct primes the two graphs have the same Sombo index and Sombor polynomial.

Keywords Sombor Index, Sombor Polynomial, Co-prime Graph, Co-prime Order Graph, Co-prime Power Order Graph.

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1 Introduction

Topological Indices are called graph invariants as they remain fixed under graph homomorphism [1], they are numerical values obtained from graphs. Such indices help in determining the structure of a molecular graph and this could aid in drug design and prediction of chemical property of molecules [2]. A graph $\Upsilon(V, E)$ is an ordered pair consisting of set of vertices, V and set of edges, E, with the elements of E linking elements of V. Structural representation of molecules can be seen via graphs, where the edges of the graph represents chemical bonds while the vertices stands for atoms.

Topological indices also play important roles in providing insights into characteristics of molecules and atom-atom interactions [3], as such there are different types of topological indices. The topological index under consideration in this paper is the Sombor index introduced by Gutman in 2021 [4], it is a degree-based topological index which has to do with geometrical approach (i.e using Euclidean metric). It involves getting the square-root of the sum of the squares of the degrees of adjacent vertices. Gutman [4] was able to generalize a formula for computing the Sombor index of some special graphs; like the star graph, path graph, and complete graph. For terminologies on graphs we see [5, 6]. Reti and Doslic [7] stated that for a connected bipartite graph, the Sombor index is an integer. What followed recently was the work of Khasraw et al. [8] where they introduced the Sombor polynomial for graphs which is an idea derived from the definition of Sombor index. The graph they used was the non-commuting graph. Pratama et al. [9] in 2024 generalized the Sombor index of the power graph of the groups of integers modulo n and that of the dihedral group, D_n, where $n = p^k, k \in \mathbb{Z}^+$. In 2025 Alimon et al. [10] looked at the Sombor index and Sombor polynomial of the power graphs for dihedral groups, generalized quaternion, and quasi-dihedral groups.

In this paper, we studied and generalized the formulae for the Sombor index and Sombor polynomial of the co-prime graphs for some finite groups. The co-prime order graph and the co-prime power order graph are also considered, both are supergraps of the co-prime graph with the latter a generalization of the former.

First section of the paper introduces the work, second section has to do with some existing results and other preliminaries needed, results and discussion are presented in third section while the conclusion is given in the fouth section.

2 Methodology

In this paper we denote the graph of a finite group by Υ, the Sombor index of Υ by $S_I(\Upsilon)$ and $S_P(\Upsilon; x)$ for the Sombor polynomial, and τ represents the size of a graph. We first consider the following definitions.

Definition 2.1. [4] For a graph Υ, the Sombor index of Υ is defined as $S_I(\Upsilon) = \sum_{\theta\lambda\, \in E(\Upsilon)} \sqrt{d_\theta^2 + d_\lambda^2}$, with d_θ and d_λ representing the degrees of respective vertices θ and λ in Υ.

Definition 2.2. [8] For a graph Υ, the Sombor polynomial is given by $S_P(\Upsilon; x) = \sum_{\theta\lambda\, \in E(\Upsilon)} \frac{1}{\sqrt{d_\theta^2 + d_\lambda^2}} x^{{d_\theta}^2 + {d_\lambda}^2}$, where d_θ and d_λ are the respective degrees of vertices θ and λ in Υ.

Gutman [4] gave a general formula for computing the Sombor index for some special graphs like the star graph (S_τ), path graph (P_τ) and a complete graph (K_τ) each of order τ. The results are as follows

$$S_I(S_\tau) = (\tau - 1)\sqrt{(\tau - 1)^2 + 1},$$
$$S_I(P_\tau) = 2\sqrt{5} + 2(\tau - 3)\sqrt{2}, \quad for\ \tau \geq 3,$$
$$S_I(K_\tau) = \frac{\tau(\tau - 1)^2}{\sqrt{2}}.$$

Definition 2.3. [5] Cycle Graph: A cycle graph is a regular graph of degree 2 denoted by C_τ.

Definition 2.4. [5] Wheel Graph: A wheel graph is gotten from a cycle graph by linking vertices of the cycle $C_{\tau-1}$ to a new vertex. The graph is denoted by W_τ with each of the $\tau - 1$ vertices of degree 3 while the added vertex is of degree $\tau - 1$ making the size to be $|E(W_\tau)| = 2(\tau - 1)$.

Next are some established results on the Sombor index of a regular graph as well as its Sombor polynomial.

Theorem 2.5. *[8] Let Υ be a $\beta-$regular graph with β as the degree of each vertex in Υ and m representing size of the graph. The Sombor index of Υ is given by; $S_I(\Upsilon) = \sqrt{2}\beta m$.*

Theorem 2.6. *[8] Let Υ be a $\beta-$regular graph, with β as the degree of each vertex in Υ and m representing size of the graph. The Sombor polynomial of Υ, $S_P(\Upsilon, x) = \frac{m}{\sqrt{2}\beta} x^{2\beta^2}$.*

We then look at the definition of co-prime graph as stated by Ma et al. [11]

Definition 2.7. [11] Co-prime Graph: Co-prime graph for a finite group G is a graph that has all elements of G as vertex set with two vertices say θ and λ adjacent in the graph whenever their orders are relatively prime i.e $(|\theta|, |\lambda|) = 1$.

They affirm that the graph is simple, undirected and connected. Some important result from their work which are needed follows;

Proposition 2.8. *[11] For a group G of order greater than 2, the singleton $\{e\}$ forms a dominating set. The domination number of the graph is 1 and $d_e = |G| - 1$.*

Theorem 2.9. *[11] The co-prime graph for a finite group G is regular if and only if G is isomorphic to $\mathbb{Z}_2$.*

Corollary 2.10. *[11] The co-prime graph of a group G with order of G greater than 2 is not a complete graph.*

Theorem 2.11. *[11] Let G be a group of order τ, then the co-prime graph for G is isomorphic to $K_{1,(\tau-1)}$ if and only if G is a $p-$group for some p a prime number.*

Theorem 2.12. *[11] For a finite group G, G is not a $p-$group if and only if the co-prime graph for G is not a bipartite graph.*

Theorem 2.13. *[11] For G a non-cyclic group of order $\gamma\ \delta$ with γ and δ distinct primes, the co-prime graph of G is a complete 3-partite graph.*

Theorem 2.14. *[11] For a dihedral group, D_τ, with τ odd, the degree of vertices in the co-prime graph for D_τ is given by;*

i. $d_{a^i} = \tau,\ \ 1 \leq i \leq \tau,$

ii. $d_{a^ib} \geq \tau,\ \ 1 \leq i \leq \tau.$

Theorem 2.15. *[11] Let τ be an odd prime, the co-prime graph for D_τ is isomorphic to $K_{1,(\tau-1),\tau}$.*

Theorem 2.16. *[11] For a dihedral group D_τ, where $\tau = 2^k$, $k \in \mathbb{Z}^+$. The co-prime graph for D_τ is isomorphic to $K_{1,2^k-1}$.*

In 2019, Banerjee [12] modified the definition of the co-prime graph as given by Ma et al. [11]. He used same vertex set in such away that two distinct vertices $\theta, \lambda \in \Upsilon$ are linked by an edge whenever $(|\theta|, |\lambda|) = p^r$, $r \in \{0, 1\}$ and p prime. This new version of the graph is called co-prime order graph. It is obvious that the co-prime graph is a subgraph of the co-prime order graph; Banerjee showed that the graph is connected with diameter at most 2. Recently Mushatet and Talebi [14] defined the co-prime power order graph for finite groups; it has all the elements of a group as vertex set and two distinct vertices θ and λ are adjacent in the graph if and only if $(|\theta|, |\lambda|) = p^r$, with p prime and $r \in \mathbb{Z}^+ \cup \{0\}$. The co-prime power order graph is a generalization of the co-prime order graph.

Theorem 2.17. *[12] The co-prime order graph has a domination number of 1 and each element θ of a group forms a dominating set of the graph if and only if $|\theta| = 1$ or a prime number.*

Theorem 2.18. *[12] The co-prime order graph for a group G is complete if and only if G has no element of composite order.*

Theorem 2.19. *[12] Let G be a cyclic group, the co-prime order graph of G is a complete graph if and only if the order is prime.*

Below is a corollary to the theorem above.

Corollary 2.20. *[12] For a dihedral group D_τ, $\tau > 2$, the co-prime order graph is complete if and only if τ is prime.*

Theorem 2.21. *[13] If $|\theta|$ is a prime number, $d_\theta = \tau - 1$ in any co-prime order graph for a group of order τ.*

Next theorem is on co-prime power order graph.

Theorem 2.22. *[14] For a finite group G the co-prime power order graph is a complete graph if $|G| > 2$ and the order of every non-trivial element of the group is of prime power order.*

The strategy employed in proving the results of this work lies on using the degree of vertices of graphs and the existing defintions of Sombor index and Sombor polynomial.

3 Results and Discussion

We first look at the Sombor polynomials of some special graphs whose Sombor index are given in [4]. As stated ealier we denote the number of vertices in a graph by $\tau \in \mathbb{N}$ and m the number of edges, which makes the order of a group G to be τ since all the elements of a group serve as the vertex set.

Theorem 3.1. *The Sombor polynomial of a star graph, $S_P(S_\tau; x)$, is a polynomial of the form $\frac{(\tau-1)}{\sqrt{(\tau-1)^2+1}}x^{(\tau-1)^2+1}$.*

Proof. By the fact that the vertex set of a star graph is partitioned into two sets say A and B with A a singleton, making $d_\theta = \tau - 1$ for $\theta \in A$ and $d_\lambda = 1$, for every $\lambda \in B$. The size of a star graph $|E| = \tau - 1$, by Definition 2.2;

$$S_P(S_\tau; x) = \sum_{\theta\lambda \,\in\, E(S_\tau)} \frac{1}{\sqrt{d_\theta^2 + d_\lambda^2}} x^{{d_\theta}^2 + {d_\lambda}^2} = \frac{|E|}{\sqrt{(\tau-1)^2+1}} x^{(\tau-1)^2+1} = \frac{(\tau-1)}{\sqrt{(\tau-1)^2+1}} x^{(\tau-1)^2+1}. \qquad \square$$

Theorem 3.2. *Let P_τ be a path graph of order τ, the Sombor polynomial of P_τ, $S_P(P_\tau; x) = \frac{2}{\sqrt{5}}x^5 + \frac{(\tau-3)}{2\sqrt{2}}x^8$.*

Proof. The path graph has $\tau - 3$ edges linking vertices of degree 2 with the remaining two edges linking an end vertex and a vertex of degree 2, by Definition 2.2 the Sombor polynomial;

$$S_\tau(P_\tau, x) = \sum_{\theta\lambda \,\in\, E(P_\tau)} \frac{1}{\sqrt{d_\theta^2 + d_\lambda^2}} x^{{d_\theta}^2 + {d_\lambda}^2} = \frac{2}{\sqrt{1^2+2^2}} x^{1+2^2} + \frac{(\tau-3)}{\sqrt{2^2+2^2}} x^{2^2+2^2} = \frac{2}{\sqrt{5}}x^5 + \frac{(\tau-3)}{2\sqrt{2}}x^8. \qquad \square$$

Next is the Sombor index and Sombor polynomial for a wheel graph.

Theorem 3.3. *For a wheel graph W_τ of order τ, the Sombor index of W_τ, $S_I(W_\tau) = (\tau - 1)[3\sqrt{2} + \sqrt{(\tau-1)^2 + 9}]$.*

Proof. A wheel graph has $2(\tau - 1)$ edges with $\tau - 1$ joining vertices of degree 3 just like $\tau - 1$ edges joins the added vertex and all remaining vertices. From Definitions 2.1 and 2.4 the Sombor index of W_τ becomes;

$$S_I(W_\tau) = \sum_{\theta\lambda \,\in\, E(W_\tau)} \sqrt{d_\theta^2 + d_\lambda^2} = (\tau-1)\sqrt{2(3^2)} + (\tau-1)\sqrt{(\tau-1)^2 + 3^2}$$

$$= (\tau-1)[3\sqrt{2} + \sqrt{(\tau-1)^2 + 9}]. \qquad \square$$

Theorem 3.4. *The Sombor polynomial of a wheel graph* W_τ *is a polynomial of the form;* $S_P(W_\tau;x) = (\tau-1)[\frac{1}{3\sqrt{2}}x^{18} + \frac{1}{\sqrt{(\tau-1)^2+9}}x^{(\tau-1)^2+9}]$.

Proof. By Definition 2.4 there are $\tau-1$ vertices each of degree 3 in a wheel graph while the single added vertex is of degree $\tau-1$, with the size of the graph as $2(\tau-1)$. The number of edges connecting the single vertex with the other vertices of degree 3 is $(\tau-1)$ as such $\tau-1$ edges also link vertices of degree 3. Applying the degrees of respective vertices to Definition 2.2 gives;

$$S_P(W_\tau;x) = \sum_{\theta\lambda\,\in\,E(W_\tau)} \frac{1}{\sqrt{d_\theta^2+d_\lambda^2}}x^{{d_\theta}^2+{d_\lambda}^2} = \frac{(\tau-1)}{\sqrt{2(3^2)}}x^{2(3^2)} + \frac{(\tau-1)}{\sqrt{(\tau-1)^2+3^2}}x^{(\tau-1)^2+3^2}$$

$$= (\tau-1)[\tfrac{1}{3\sqrt{2}}x^{18} + \tfrac{1}{\sqrt{(\tau-1)^2+9}}x^{(\tau-1)^2+9}].$$ □

The following set of theorems are on the co-prime graphs.

Theorem 3.5. *The Sombor polynomial for co-prime graph,* Υ*, of a group of order 2 is a monomial of degree 2 given by;* $S_P(\Upsilon;x) = \frac{1}{\sqrt{2}}x^2$.

Proof. It is easily seen from Theorem 2.9 that a group of order 2 is a $\beta-$regular graph with $\beta = 1$ and of size 1, making the Sombor index to be just $\sqrt{2}$. Applying Theorem 2.6 with $\beta = 1$ and $m = 1$ gives the result. □

The co-prime graph for a p-group is a star graph as seen in Theorem 2.11, so the Sombor polynomial for a $p-$group is given in Theorem 3.1. Likewise, for a dihedral group, D_τ, where $\tau = 2^k, k \in \mathbb{Z}^+$ the co-prime graph as shown in Theorem 2.16 has same Sombor index and Sombor polynomial as a star graph.

Lemma 3.6. *For a non-cyclic group G of order* $\gamma\delta$*, where* γ *and* δ *are distinct primes. The degree of a vertex in the graph is given by*

$$deg_\theta = \begin{cases} |G|-1, & \text{if } |\theta| = 1, \\ |G|-\rho, & \text{if } |\theta| = \gamma, \\ 1+\rho, & \text{if } |\theta| = \delta, \end{cases}$$

where ρ *is the order of the set consisitng of elements of order* γ.

Proof. Since G is non-cyclic and $|G| = \gamma\,\delta$, elements of G could be of order 1, γ or δ. From the definition of the graph $\theta, \lambda \in G$ are adjacent if and only if $(|\theta|, |\lambda|) = 1$.

Case 1: The graph is a complete 3-partite graph as seen in Theorem 2.13 , let the partite sets of the graph be A, B and C. Let $A = \{e\}$, this implies that A is the dominating set making $deg_e = |G|-1$ as seen in Proposition 2.8.

Case 2: Let $|B| = \rho$, from Definition 2.7 each $\theta \in B$ is adjacent to every element in G except those in B. This implies that $deg_\theta = |G|-\rho$.

Case 3: This is also straight forward. □

The next two results are on the Sombor index and Sombor polynomial for a non-cyclic group G of order a product of two distinct primes.

Theorem 3.7. *Let* $\Upsilon(G)$ *be the co-prime graph of a finite non-cyclic group G of order* τ*, with* τ *a product of two distinct prime numbers. Then, the Sombor index of* Υ *is given by;* $S_I(\Upsilon(G) =$
$\rho\sqrt{(\tau-1)^2+(\tau-\rho)^2} + [\tau-(\rho+1)]\sqrt{(\tau-1)^2+(1+\rho)^2} + [\rho(\tau-(\rho+1))]\sqrt{(\tau-\rho)^2+(1+\rho)^2}$*, where* ρ *represents the order of one of the non-trivial partite set from the vertex set of* $\Upsilon(G)$.

Proof. Let the order of a non-cyclic group G be $\tau = \gamma\delta$ where γ and δ are distinct primes, from Theorem 2.13 the co-prime graph of G is a complete 3-partite graph. Let A, B and C be the partite sets with each an independent set, let $|A| = \rho$, then $|B| = \tau-(\rho+1)$ since a partite set contains only the identity element based on the definition of the co-prime graph as seen in Definition 2.7. From Theorem 2.8 $d_e = \tau-1$, since each partite set is an independent set we have; $d_\theta = \tau-\rho$, $\forall\ \theta \in A$ and $d_\theta = 1+\rho, \forall\ \theta \in B$.

Note that the number of edges between the identity element and elements of sets A and B equals $|A|$ and $|B|$ respectively, while that between elements of sets A and B equals $|A||B|$. By Definition 2.1,

$$S_I(\Upsilon(G)) = \sum_{\theta\lambda\,\in\,E(\Upsilon(G))} \sqrt{d_\theta^2+d_\lambda^2}$$

$$= \rho\sqrt{(\tau-1)^2+(\tau-\rho)^2} + [\tau-(\rho+1)]\sqrt{(\tau-1)^2+(1+\rho)^2} + [\rho(\tau-(\rho+1))]\sqrt{(\tau-\rho)^2+(1+\rho)^2}.$$ □

Theorem 3.8. *Let* $\Upsilon(G)$ *be the co-prime graph of a finite non-cyclic group G of order* τ*, with* τ *a product of two distinct prime numbers. Then, the Sombor polynomial of* $\Upsilon(G)$ *is a polynomial of the form;* $S_P(\Upsilon(G),x) =$
$\frac{\rho}{\sqrt{(\tau-1)^2+(\tau-\rho)^2}}x^{(\tau-1)^2+(\tau-\rho)^2} + \frac{[\tau-(1+\rho)]}{\sqrt{(\tau-1)^2+(1+\rho)^2}}x^{(\tau-1)^2+(1+\rho)^2} + \frac{[\rho(\tau-(1+\rho))]}{\sqrt{(\tau-\rho)^2+(1+\rho)^2}}x^{(\tau-\rho)^2+(1+\rho)^2}$.

Proof. Applying Lemma 3.6 to Definition 2.2 gives the result. $\square$

For τ an odd prime, the co-prime graph for dihedral group, D_τ, is a complete 3-partite graph $(K_{1,(\tau-1),\tau})$ as shown in Theorem 2.15 as such the Sombor index and Sombor polynomial are similar to what is obtained in Theorems 3.7 and 3.8. The co-prime graph for a cyclic group G of order $\gamma\,\delta$ is not a complete graph since there are elements of composite order $(\gamma\,\delta)$ which are not adjacent to those of prime order. The next lemma gives the degree of each vertex in a co-prime graph for a cyclic group of order $\gamma\,\delta$.

Lemma 3.9. *For a cyclic group G of order $\gamma\delta$ with γ and δ distinct primes, the degree of vertices of the co-prime graph for G is given by;*

$$deg_\theta = \begin{cases} |G|-1, & \textit{if} \quad |\theta| = 1 \\ \gamma, & \textit{if} \quad |\theta| = \delta \\ \delta, & \textit{if} \quad |\delta| = \gamma \\ 1, & \textit{otherwise}. \end{cases}$$

Proof. The proof is straight forward since $(|\gamma|, |\delta|) = 1$ and an element θ of order $\gamma(\textit{or } \delta)$ would be linked to other elements of order $\delta(\textit{or } \gamma)$ and the identity element which is $\phi(\delta) + 1 = \delta$(or $\phi(\gamma) + 1 = \gamma$) in number, with ϕ the Euler's phi function. $\square$

Theorem 3.10. *Let G be a cyclic group of order $\gamma\delta$ with γ and δ distinct primes, the Sombor index for the co-prime graph for G is* $S_I(\Upsilon(G)) = (\delta-1)\sqrt{(\gamma\delta-1)^2+\gamma^2} + (\gamma-1)\sqrt{(\gamma\delta-1)^2+\delta^2} + (\gamma-1)(\delta-1)[\sqrt{(\gamma\delta-1)^2+1} + \sqrt{\gamma^2+\delta^2}]$.

Proof. From Definition 2.1 we have $S_I(\Upsilon(G)) = \sum_{\theta\lambda\in E(\Upsilon(G))} \sqrt{d_\theta^2 + d_\lambda^2} = \sqrt{d_\theta^2 + d_\lambda^2} \times \phi(\theta)\phi(\lambda)$, where $\theta, \lambda \in E(\Upsilon(G))$, $(|\theta|, |\lambda|) = 1$ and ϕ the Euler's phi function. Applying Lemma 3.9 gives;

$S_I(\Upsilon(G)) = \sqrt{(\gamma\,\delta-1)^2+\gamma^2} \times \phi(\delta) + \sqrt{(\gamma\,\delta-1)^2+\delta^2} \times \phi(\gamma) + \sqrt{(\gamma\,\delta-1)^2+1^2} \times \phi(\gamma\,\delta) + \sqrt{\gamma^2+\delta^2} \times \phi(\gamma)\phi(\delta)$

$= (\delta-1)\sqrt{(\gamma\,\delta-1)^2+\gamma^2} + (\gamma-1)\sqrt{(\gamma\,\delta-1)^2+\delta^2} + (\gamma-1)(\delta-1)[\sqrt{(\gamma\,\delta-1)^2+1} + \sqrt{\gamma^2+\delta^2}]$. $\square$

Next is the corresponding Sombor polynomial for the cyclic group of order $\gamma\,\delta$.

Theorem 3.11. *The Sombor polynomial for the co-prime graph $\Upsilon(G)$ of a cyclic group G of order $\tau = \gamma\,\delta$ is a polynomial of the form;*

$S_p(\Upsilon(G); x) = \frac{\delta-1}{\sqrt{(\tau-1)^2+\gamma^2}} x^{(\tau-1)^2+\gamma^2} + \frac{\gamma-1}{\sqrt{(\tau-1)^2+\delta^2}} x^{(\tau-1)^2+\delta^2} + (\gamma-1)(\delta-1)[\frac{1}{\sqrt{(\tau-1)^2+1}} x^{(\tau-1)^2+1} + \frac{1}{\sqrt{\gamma^2+\gamma^2}} x^{\gamma^2+\delta^2}]$.

Proof. We apply Lemma 3.9 to Definition 2.2 to get

$$S_P(\Upsilon(G), x) = \sum_{\theta\ \lambda\in E(\Upsilon(G))} \frac{1}{\sqrt{d_\theta^2 + d_\lambda^2}} x^{d_\theta{}^2 + d_\lambda{}^2}$$

$= \frac{1}{\sqrt{(\tau-1)^2+\gamma^2}} x^{(\tau-1)^2+\gamma^2} \times \phi(\delta) + \frac{1}{\sqrt{(\tau-1)^2+\delta^2}} x^{(\tau-1)^2+\delta^2} \times \phi(\gamma) + \frac{1}{\sqrt{(\tau-1)^2+1^2}} x^{(\tau-1)^2+1^2} \times \phi(\gamma\delta) + \frac{1}{\sqrt{\gamma^2+\delta^2}} x^{\gamma^2+\delta^2} \times \phi(\gamma)\phi(\delta)$

$= \frac{\delta-1}{\sqrt{(\tau-1)^2+\gamma^2}} x^{(\tau-1)^2+\gamma^2} + \frac{\gamma-1}{\sqrt{(\tau-1)^2+\delta^2}} x^{(\tau-1)^2+\delta^2} + (\gamma-1)(\delta-1)[\frac{1}{\sqrt{(\tau-1)^2+1}} x^{(\tau-1)^2+1} + \frac{1}{\sqrt{\gamma^2+\delta^2}} x^{\gamma^2+\delta^2}]$. $\square$

The results in Theorems 3.10 and 3.11 also applies to a cyclic subgroup of a dihedral group, D_τ, where $|\langle a\rangle| = \gamma\,\delta$, γ and δ distinct primes.

Sombor Polynomials of The Co-prime Order Graph

Note that for the co-prime order graph [12] which is a supergraph of the co-prime graph defined by Ma et al. [11], the following are clear

Remark 3.12.

1. The co-prime order graph for a non-cyclic group of order $\gamma\,\delta$ where γ and δ are distinct primes is a complete graph.
2. For a dihedral group $D_\tau, \tau > 2$ and τ is prime; the co-prime order graph is a complete graph.

It is easy to observe that the graphs in Remark 3.12 are all complete graphs since two vertices θ and λ are adjacent if and only if $(|\theta|, |\lambda|) = p^r, r \in \{0, 1\}$. The Sombor index and Sombor polynomial for the each graph are similar results to those of Theorem 2.5 and Theorem 2.6. For a $p-$group we don't have a regular graph as seen in the following lemma.

Lemma 3.13. *Let G be a $p-$group, the co-prime order graph for G has the degree of vertices as;*
$$deg_\theta = \begin{cases} |G|-1, & \text{if } |\theta| = 1 \text{ or } p \\ 1+\phi(p), & \text{otherwise,} \end{cases}$$
where ϕ is the Euler's phi function.

Proof. Case 1 is clear from Theorem 2.17.
Case 2: Let $|G| = p^k, k \in \mathbb{Z}^+$, two distinct elements $\theta, \lambda \in G$ with $|\theta| = p^i$ and $|\lambda| = p^{i+m}$ where $i = 2, 3, \cdots, k$ and $m \geq 1$ are not adjacent in the graph since $(p^i, p^{i+m}) = p^i$. The elements of order p^i are only adjacent to the identity element and those elements of order p. ☐

We then look at the Sombor index and Sombor polynomial of the co-prime order graph for a $p-$group.

Theorem 3.14. *The Sombor index for the co-prime order graph of a $p-$group G of order τ is given by; $S_I(\Upsilon(G)) = (p-1)\sqrt{2(\tau-1)^2} + 2(\tau-p)\sqrt{(\tau-1)^2+p^2}$.*

Proof. Applying Lemma 3.13 to Definition 2.1 gives;
$$S_I(\Upsilon(G)) = \sum_{\theta\ \lambda \in E(\Upsilon(G))} \sqrt{d_\theta^2 + d_\lambda^2}$$
$$= \sqrt{(\tau-1)^2 + (\tau-1)^2} \times \phi(p) + \sqrt{(\tau-1)^2+p^2} \times (\tau - (\phi(p)+1)) + \sqrt{(\tau-1)^2+p^2} \times (\tau - (\phi(p)+1))$$
$$= (p-1)\sqrt{2(\tau-1)^2} + 2(\tau-p)\sqrt{(\tau-1)^2+p^2}.$$
☐

Theorem 3.15. *Let G be a $p-$group, the Sombor polynomial for the co-prime order graph of G is a polynomial of the form; $\frac{p-1}{\sqrt{2(\tau-1)^2}}x^{2(\tau-1)^2} + \frac{2(\tau-p)}{\sqrt{(\tau-1)^2+p^2}}x^{(\tau-1)^2+p^2}$.*

Proof. Using the degrees of vertices of the co-prime order graph for a $p-$group as stated in Lemma 3.13, from Definition 2.2 we get the result as
$$S_P(\Upsilon(G), x) = \sum_{\theta\lambda\ \in E(\Upsilon(G))} \frac{1}{\sqrt{d_\theta^2 + d_\lambda^2}} x^{{d_\theta}^2 + {d_\lambda}^2}$$
$$= \frac{\phi(p)}{\sqrt{(\tau-1)^2+(\tau-1)^2}}x^{(\tau-1)^2+(\tau-1)^2} + \frac{1}{\sqrt{(\tau-1)^2+p^2}}x^{(\tau-1)^2+p^2} \times (\tau - (\phi(p)+1)) + \frac{1}{\sqrt{(\tau-1)^2+p^2}}x^{(\tau-1)^2+p^2} \times (\tau - (\phi(p)+1))$$
$$= \frac{p-1}{\sqrt{2(\tau-1)^2}}x^{2(\tau-1)^2} + \frac{2(\tau-p)}{\sqrt{(\tau-1)^2+p^2}}x^{(\tau-1)^2+p^2}.$$
☐

From Theorem 2.18 we can say that a cyclic group of order $\gamma\delta$ with γ and δ distinct primes has a co-prime order graph which is not a regular graph, i.e the vertices of the graph have different degrees since G has elements of composite order $(\gamma\delta)$. The following lemma shows the degree of respective vertices.

Lemma 3.16. *Let $\Upsilon(G)$ be the co-prime order graph for a cyclic group of order $\gamma\delta$, with γ and δ distinct primes. The degree of vertices of $\Upsilon(G)$ is given as;*
$$deg_\theta = \begin{cases} |G|-1, & \text{if } |\theta| = 1 \text{ or prime} \\ |G|-\phi(\gamma\delta), & \text{otherwise,} \end{cases}$$
where ϕ denotes the Euler's phi function.

Proof. The proof is clear from Theorems 2.17 and 2.21; and also for the fact that elements of order $\gamma\delta$ are adjacent to every other element except those of order $\gamma\delta$. ☐

We use Lemma 3.16 to get the next Sombor index and Sombor plynomial.

Theorem 3.17. *Let $\Upsilon(G)$ be the co-prime order graph of a cyclic group G of order $\gamma\delta$ with γ and δ distinct primes, then the Sombor index of $\Upsilon(G)$ is given by;*
$$\sqrt{2}(\gamma\delta-1)^2 + \sqrt{(\gamma\delta-1)^2 + (\gamma+\delta-1)^2}(\gamma\delta(\gamma+\delta) + 2(\gamma+\delta) - (\gamma^2+\delta^2) - 3\gamma\delta - 1).$$

Proof. We apply Lemma 3.16 to Definition 2.1 as shown below;
$$S_I(\Upsilon(G)) = \sum_{\theta\lambda \in E(\Upsilon(G))} \sqrt{d_\theta^2 + d_\lambda^2}$$
$$= \sqrt{(\gamma\delta-1)^2 + (\gamma\delta-1)^2} \times \phi(\gamma) + \sqrt{(\gamma\delta-1)^2 + (\gamma\delta-1)^2} \times \phi(\delta) + \sqrt{(\gamma\delta-1)^2 + (\gamma\delta-\phi(\gamma\delta))^2} \times \phi(\gamma\delta) +$$
$$\sqrt{(\gamma\delta-1)^2 + (\gamma\delta-1)^2} \times \phi(\gamma)\phi(\delta) + \sqrt{(\gamma\delta-1)^2 + (\gamma\delta-\phi(\gamma\delta))^2} \times \phi(\gamma)\phi(\gamma\delta) + \sqrt{(\gamma\delta-1)^2 + (\gamma\delta-\phi(\gamma\delta))^2} \times \phi(\delta)\phi(\gamma\delta)$$

$= \sqrt{2(\gamma\delta-1)^2}\times(\gamma-1)+\sqrt{2(\gamma\delta-1)^2}\times(\delta-1)+\sqrt{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}\times(\gamma-1)(\delta-1)+\sqrt{2(\gamma\delta-1)^2}\times(\gamma-1)(\delta-1)+\sqrt{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}\times(\gamma-1)^2(\delta-1)+\sqrt{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}\times(\gamma-1)(\delta-1)^2$

$= \sqrt{2(\gamma\delta-1)^2}[(\gamma-1)+(\delta-1)+(\gamma-1)(\delta-1)]+\sqrt{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}[(\gamma-1)(\delta-1)+(\gamma-1)^2(\delta-1)+(\gamma-1)(\delta-1)^2]$
$= \sqrt{2}(\gamma\delta-1)^2+\sqrt{(\gamma\delta-1)^2+(\gamma+\delta-1)^2}(\gamma\delta(\gamma+\delta)+2(\gamma+\delta)-(\gamma^2+\delta^2)-3\gamma\delta-1).$ ☐

Theorem 3.18. *The Sombor polynomial for the co-prime order graph* $(\Upsilon(G))$ *for a cyclic group of order* $\gamma\delta$ *with* γ *and* δ *distinct primes is given by;*
$= \frac{1}{\sqrt{2}}x^{2(\gamma\delta-1)^2}+\frac{(\gamma\delta(\gamma+\delta)+2(\gamma+\delta)-(\gamma^2+\delta^2)-3\gamma\delta-1)}{\sqrt{(\gamma\delta-1)^2+(\gamma+\delta-1)^2}}x^{(\gamma\delta-1)^2+(\gamma+\delta-1)^2}.$

Proof. We apply Lemma 3.16 to Definition 2.2 to obtain the Sombor polynomial; having in mind that the number of edges between the vertices can be obtained using the Euler's phi function.

$$S_P(\Upsilon(G),x) = \sum_{\theta\lambda\in E(\Upsilon(G))}\frac{1}{\sqrt{d_\theta^2+d_\lambda^2}}x^{{d_\theta}^2+{d_\lambda}^2}$$

$$= \frac{1}{\sqrt{(\gamma\delta-1)^2+(\gamma\delta-1)^2}}x^{2(\gamma\delta-1)^2}\times\phi(\gamma)+\frac{1}{\sqrt{(\gamma\delta-1)^2+(\gamma\delta-1)^2}}x^{2(\gamma\delta-1)^2}\times\phi(\delta)+$$
$$\frac{1}{\sqrt{(\gamma\delta-1)^2+(\gamma\delta-\phi(\gamma\delta))^2}}x^{(\gamma\delta-1)^2+(\gamma\delta-\phi(\gamma\delta))^2}\times\phi(\gamma\delta)+\frac{1}{\sqrt{(\gamma\delta-1)^2+(\gamma\delta-1)^2}}x^{2(\gamma\delta-1)^2}\times\phi(\gamma)\phi(\delta)+$$
$$\frac{1}{\sqrt{(\gamma\delta-1)^2+(\gamma\delta-\phi(\gamma\delta))^2}}x^{(\gamma\delta-1)^2+(\gamma\delta-\phi(\gamma\delta))^2}\times\phi(\gamma)\phi(\gamma\delta)+\frac{1}{\sqrt{(\gamma\delta-1)^2+(\gamma\delta-\phi(\gamma\delta))^2}}x^{(\gamma\delta-1)^2+(\gamma\delta-\phi(\gamma\delta))^2}\times\phi(\delta)\phi(\gamma\delta)$$
$$= \frac{(\gamma-1)}{\sqrt{2(\gamma\delta-1)^2}}x^{2(\gamma\delta-1)^2}+\frac{(\delta-1)}{\sqrt{2(\gamma\delta-1)^2}}x^{2(\gamma\delta-1)^2}+\frac{(\gamma-1)(\delta-1)}{\sqrt{(\gamma\delta-1)^2+[\gamma\delta-(\gamma-1)(\delta-1)]^2}}x^{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}+\frac{(\gamma-1)(\delta-1)}{\sqrt{2(\gamma\delta-1)^2}}x^{2(\gamma\delta-1)^2}+$$
$$\frac{(\gamma-1)^2(\delta-1)}{\sqrt{(\gamma\delta-1)^2+[\gamma\delta-(\gamma-1)(\delta-1)]^2}}x^{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}+$$
$$\frac{(\gamma-1)(\delta-1)^2}{\sqrt{(\gamma\delta-1)^2+[\gamma\delta-(\gamma-1)(\delta-1)]^2}}x^{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}$$

$$= [\frac{(\gamma-1)}{\sqrt{2(\gamma\delta-1)^2}}+\frac{(\delta-1)}{\sqrt{2(\gamma\delta-1)^2}}+\frac{(\gamma-1)(\delta-1)}{\sqrt{2(\gamma\delta-1)^2}}]x^{2(\gamma\delta-1)^2}+$$
$$[\frac{(\gamma-1)(\delta-1)}{\sqrt{(\gamma\delta-1)^2+[\gamma\delta-(\gamma-1)(\delta-1)]^2}}+\frac{(\gamma-1)^2(\delta-1)}{\sqrt{(\gamma\delta-1)^2+[\gamma\delta-(\gamma-1)(\delta-1)]^2}}+\frac{(\gamma-1)(\delta-1)^2}{\sqrt{(\gamma\delta-1)^2+[\gamma\delta-(\gamma-1)(\delta-1)]^2}}]x^{(\gamma\delta-1)^2+(\gamma\delta-(\gamma-1)(\delta-1))^2}$$

$$= \frac{(\gamma\delta-1)}{\sqrt{2(\gamma\delta-1)^2}}x^{2(\gamma\delta-1)^2}+\frac{(\gamma\delta(\gamma+\delta)+2(\gamma+\delta)-\gamma^2-\delta^2-3\gamma\delta-1)}{\sqrt{(\gamma\delta-1)^2+(\gamma+\delta+1)^2}}x^{(\gamma\delta-1)^2+(\gamma+\delta+1)^2}$$
$$= \frac{1}{\sqrt{2}}x^{2(\gamma\delta-1)^2}+\frac{(\gamma\delta(\gamma+\delta)+2(\gamma+\delta)-(\gamma^2+\delta^2)-3\gamma\delta-1)}{\sqrt{(\gamma\delta-1)^2+(\gamma+\delta-1)^2}}x^{(\gamma\delta-1)^2+(\gamma+\delta-1)^2}.$$ ☐

Sombor Polynomials of The Co-prime Power Order Graph

The co-prime power order graph is a generalization of the co-prime order graph since the adjacency condition is that two distinct vertices are adjacent in the graph if and only if $(|\theta|,|\lambda|)=p^r, r\in\mathbb{Z}^+\cup\{0\}$.

Remark 3.19. From Theorem 2.22 we can say that the co-prime power order graph of each of the following groups is a complete graph.

1. $p-$group
2. The generalized quaternion, $\mathbb{Q}_{4n}$, where $n=2^m, m\in\mathbb{Z}^+$.

Theorem 3.20. *Let* Υ *be the co-prime power order graph for a* $p-$*group* G *of order* τ*, the Sombor index for the group is given by* $S_I(\Upsilon(G))=\sqrt{2}(\tau-1)m$ *with* m *the size of the graph and* $\tau>2$.

Proof. A $p-$group of order τ has the co-prime power order graph a complete graph K_τ as seen in Remark 3.19, i.e a $\beta-regular$ graph with $\beta=\tau-1$. From Theorem 2.5 we have the desired result as $S_I(\Upsilon(G))=K_\tau=\sqrt{2}(\tau-1)m$ with m representing size of the graph i.e $m=\frac{\tau(\tau-1)}{2}$. ☐

Theorem 3.21. *For a co-prime power order graph of a* $p-$*group* G *of order* τ*, where* $\tau>2$ *. The Sombor polynomial is a monomial of the form* $\frac{m}{\sqrt{2}(\tau-1)}x^{2(\tau-1)^2}$*, where* m *is the size of the graph.*

Proof. The proof is clear from Theorem 2.6 with $\beta=\tau-1$ and m size of the graph. ☐

We then look at the degree of vertices of the co-prime power order grpah for a cyclic group of order product of two primes in the following lemma.

Lemma 3.22. *Let $\Upsilon(G)$ be the co-prime power order graph of a cyclic group G of order $\gamma\delta$ with γ and δ distinct primes. The degree of vertices of $\Upsilon(G)$ is given by;*

$$deg_\theta = \begin{cases} |G|-1, & \text{if } |\theta| = 1 \ or \ prime \\ |G|-\phi(\gamma\delta), & otherwise \end{cases},$$

where ϕ is the Euler's phi function.

Proof. The result is the same as that of Lemma 3.16 hence same proof as the groups under consideration are also the same. □

From Lemm 3.22 we can say that the Sombor index and Sombor polynomial for the co-prime power order graph of a cyclic group of order the product of two distinct primes is the same as that of the co-prime order graph for a cyclic group of order product of two distinct primes as captured in Theorems 3.17 and 3.18 .

For a non-cyclic group G of order $\gamma\delta$, the co-prime power oder graph is a complete graph $K_{\gamma\delta}$, which is a $\beta - regular$ graph with $\beta = \gamma\delta - 1$. The Sombor index and Sombor polynomial is as depicted in Theorems 2.5 and 2.6 with $\beta = \gamma\delta - 1$ and $m = \frac{\gamma\delta(\gamma\delta-1)}{2}$, which is the same result for a co-prime order graph of a non-cyclic group G of order $\gamma\delta$.

Despite the fact that the co-prime graph is a subgraph of both the co-prime order graph and co-prime order power graph, it has a different Sombor index and Sombor polynomials.

4 Conclusion

The Sombor index and Sombor polynomial for the co-prime graph of $p-$group, cyclic and non-cyclic groups of order the product of two distinct primes are investiagted. For the $p-$group it has same result as a star graph. Formulae for computing the Sombor index and Sombor polynomial for two variations of the co-prime graphs are also lntroduced, i.e the co-prime order graph and the co-prime power order graph, and it is shown that they are the same for cyclic and non-cyclic groups of order product of two distinct prime numbers. The Sombor polynomials for some special graphs are also presented; the graphs are the star graph, path graph and a wheel graph.

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